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On the symmetry of simple 16-hedra

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The symmetry point group statistics for all combinatorially non-isomorphic convex simple 16-hedra (17 490 241 in total) are contributed in the paper for the first time. The most symmetrical polyhedra with 6 to 56 automorphism group orders (165 in total) are drawn in Schlegel diagrams and characterized by facet symbols and symmetry point groups.

1. Introduction

The numbers of combinatorial types of convex simple 4- to 15-hedra with given combinatorial automorphism group orders were summarized in Engel (2003). We determined the symmetry point groups of all 4- to 11- and simple 12- and 13-hedra which are isomorphous to the related automorphism groups (Voytekhovsky, 2001*a*,*b*; Voytekhovsky & Stepenshchikov, 2002*a*,*b*, 2003*a*). The symmetry point group statistics of simple 14- and 15-hedra were contributed in Voytekhovsky & Stepenshchikov (2003*b*). The variety of not simple 12-hedra was revised in Voytekhovsky & Stepenshchikov (2005). Here, we contribute the symmetry point group statistics of simple 16-hedra with Schlegel diagrams of the most symmetrical ones for the first time.

2. Generation of polyhedra

As the simple 15-hedra have already been found, we used them to generate all the simple 16-hedra by the Fedorov (1893) recurrence algorithm briefly described in Voytekhovsky (2001*b*). As in all previous cases, we generated the polyhedra as their Schlegel diagrams. This is justified by the Steinitz (every 3-connected planar graph can be realized as a polyhedron) and Mani (every combinatorial automorphism of a polyhedron is affinely realizable) theorems (Steinitz, 1922; Steinitz & Rademacher, 1934). The algorithmic procedures we used are very close to those described in detail by Engel (2003). What we need to find the simple 16-hedra is to apply α, β and γ procedures by Fedorov (*i.e.* 0-, 1- and 2-face cuts by Engel) to all simple 15-hedra. The determination of the symmetry point groups isomorphous to the related automorphism groups were done with an original computer algorithm. Briefly, we calculate the automorphism group order of a polyhedron as the number of different vertex reindexings that save its adjacency matrix. Afterwards, any such reindexing is identified as a symmetry element (plane, axis, inversion axis or center) in accordance with some rules.

3. Results and discussion

The ⟨automorphism group order⟩ and (symmetry point group) statistics of simple 16-hedra, 17 490 241 in total, are as follows: ⟨1⟩ (1) 17 411 448; ⟨2⟩ 77 415: (*m*) 67 035, (2) 9939, ($\bar{1}$) 441; ⟨3⟩ (3) 111; ⟨4⟩ 1102: (*mm*2) 871, (2/*m*) 165, (222) 46, ($\bar{4}$) 20; ⟨6⟩ (3*m*) 114; ⟨8⟩ 45: (*mmm*) 24, ($\bar{4}2m$) 21; ⟨12⟩ (23) 1; ⟨14⟩ (7*m*) 1; ⟨24⟩ ($\bar{4}3m$) 2; ⟨28⟩ ($\bar{7}m$) 1; ⟨56⟩ (14/*mmm*) 1. As in the cases of 4- to 12- and simple 13- to 15-hedra, the shapes of 1, *m*, 2 and *mm*2 symmetry point groups prevail among the simple 16-hedra with the trivial shapes forming the overwhelming majority. The most symmetrical simple 16-hedra with 6 to 56 automorphism group orders are enumerated below. To lexicographically order them, we use the [facet symbols] meaning the numbers of 3-, 4-, ..., *n*-gonal facets in a sequence.

[00,12,4] $\bar{4}3m$: 1; [00,14,02] $\bar{7}m$: 2; [02,12,002] *mmm*: 3; [0367] 3*m*: 4; [0393001] 3*m*: 5; [0448] *mmm*: 6–8, $\bar{4}2m$: 9–11; [04804] *mmm*: 12, $\bar{4}2m$: 13; [060,10] *mmm*: 14; [06343] 3*m*: 15–18; [0636001] 3*m*: 19; [064402] *mmm*: 20; [0660301] 3*m*: 21, 22; [066103] 3*m*: 23–25; [08044] *mmm*: 26, $\bar{4}2m$: 27–29; [080602] *mmm*: 30; [084004] $\bar{4}2m$: 31–33; [08420002] *mmm*: 34; [09016] 3*m*: 35; [0930031] 3*m*: 36; [0930300001] 3*m*: 37, 38; [0931003] 3*m*: 39, 40; [0,10,0042] *mmm*: 41, 42; [0,10,0204] *mmm*: 43; [0,10,040002] *mmm*: 44; [0,10,40000002] *mmm*: 45; [0,12,00004] $\bar{4}2m$: 46, $\bar{4}3m$: 47; [0,14,0000000002] 14/*mmm*: 48; [1096] 3*m*: 49; [10,12,03] 3*m*: 50; [1339] 3*m*: 51; [13633] 3*m*: 52–56; [16063] 3*m*: 57–59; [16306] 3*m*: 60; [163303] 3*m*: 61, 62; [190033] 3*m*: 63; [1903003] 3*m*: 64; [303,10] 3*m*: 65; [3090301] 3*m*: 66, 67; [309103] 3*m*: 68, 69; [3093000001] 3*m*: 70; [33073] 3*m*: 71–73; [3309001] 3*m*: 74, 75; [33316] 3*m*: 76; [333403] 3*m*: 77, 78; [3360300001] 3*m*: 79; [3361003] 3*m*: 80; [360133] 3*m*: 81, 82; [3603031] 3*m*: 83; [3604003] 3*m*: 84–86; [3630004] 3*m*: 87; [36310003] 3*m*: 88; [390100003] 3*m*: 89, 90; [400,12] $\bar{4}2m$: 91, 23: 92; [40363] 3*m*: 93, 94; [40444] *mmm*: 95, 96, $\bar{4}2m$: 97; [40606] 3*m*: 98; [406303] 3*m*: 99, 100; [408004] *mmm*: 101, $\bar{4}2m$: 102; [4090003] 3*m*: 103; [420802] *mmm*: 104; [43036] 3*m*: 105, 106; [430603] 3*m*: 107, 108; [433033] 3*m*: 109–114; [4333003] 3*m*: 115–118; [43600003] 3*m*: 119, 120; [440404] *mmm*: 121, $\bar{4}2m$: 122–125; [460006] *mmm*: 126, 127; [4600303] 3*m*: 128; [46030003] 3*m*: 129, 130; [4604000002] *mmm*: 131; [463000003] 3*m*: 132; [48000004] $\bar{4}2m$: 133; [60046] 3*m*: 134, 135; [603133] 3*m*: 136, 137; [6033300001] 3*m*: 138; [6034003] 3*m*: 139; [60610003] 3*m*: 140; [630106] 3*m*: 141, 142; [6301303] 3*m*: 143; [63040003] 3*m*: 144; [6330003001] 3*m*: 145; [633100003] 3*m*: 146, 147; [700333] 3*m*: 148–150; [7006003] 3*m*: 151; [700710000001] 7*m*: 152; [703006] 3*m*: 153; [7030303] 3*m*: 154–156; [70330003] 3*m*: 157; [7300033] 3*m*: 158, 159; [73003003] 3*m*: 160; [8004004] $\bar{4}2m$: 161; [80040202] *mmm*: 162; [80400004] $\bar{4}2m$: 163; [90013003] 3*m*: 164; [9003003001] 3*m*: 165.

The polyhedra are drawn in Schlegel diagrams with the numbers corresponding to the above list (Fig. 1). A projection is usually made along the main symmetry axis onto the orthogonal facet, if any. But it is difficult to draw the projections of 16-hedra inside the 3- or even 4-gonal facets. In some cases, we drew Schlegel diagrams into one of

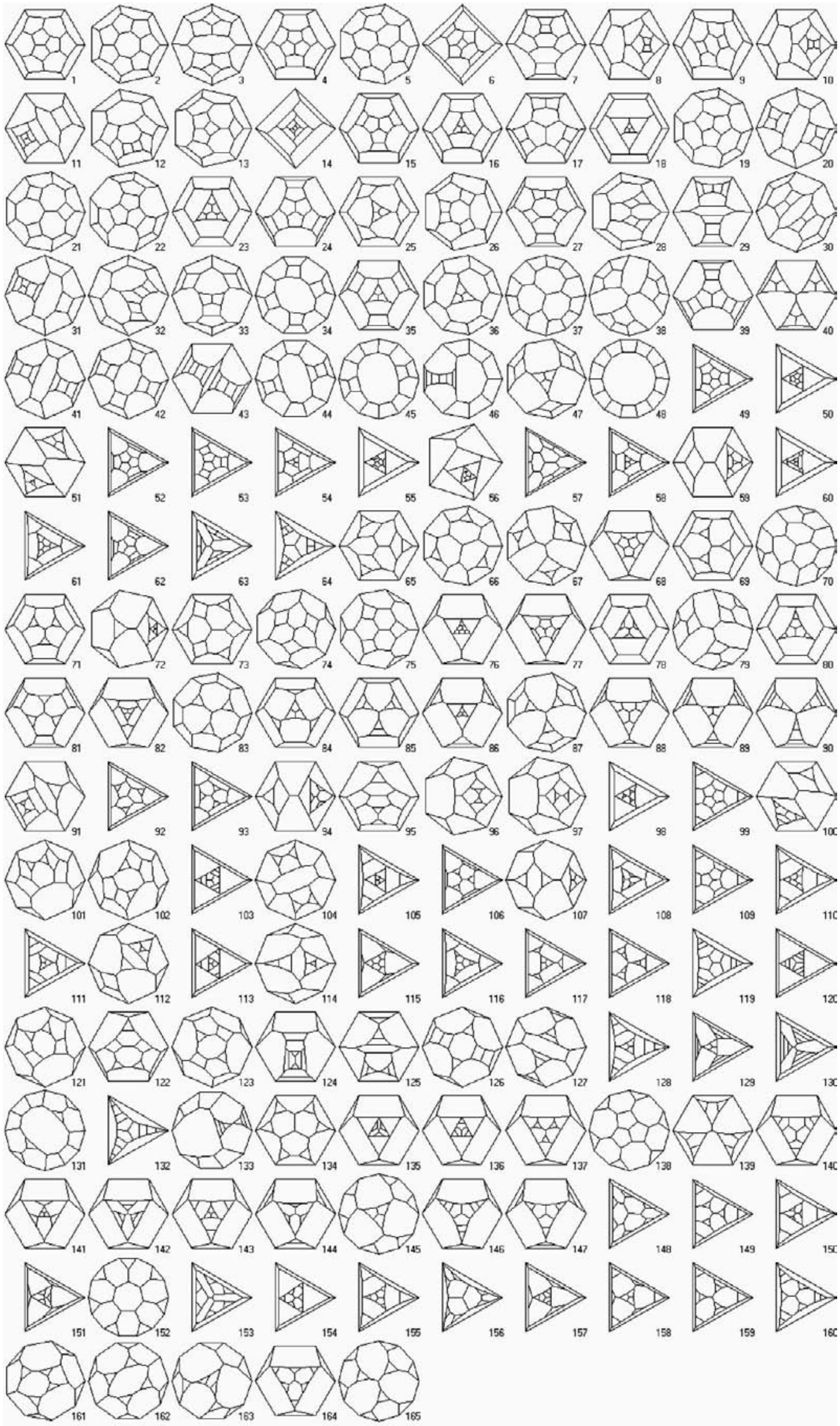

Figure 1
The most symmetrical simple 16-hedra in Schlegel diagrams.

the n-gonal facets with highest n preserving as much as possible of the symmetry. To easily understand any diagram, one should bear its facet symbol in mind and remember that only three edges meet at each vertex.

4. Conclusions

Up to now, all the varieties of 4- to 12- and simple 13- to 16-hedra have been enumerated and characterized by facet symbols and symmetry point groups. The most symmetrical shapes are drawn in Schlegel diagrams. The next steps are to generate and characterize all not simple 13- and simple 17-hedra. Unfortunately, the available computer algorithms need an enormous length of time to do these.

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